Concerning the Eddy Currents Generated by a Spontaneous Change of Magnetization

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While studying several processes, whereby ferromagnetic materials, such as iron, are subjected to the application of a periodic or abrupt change in magnetic field, eddy currents arise, which decrease the rate of demagnetization within the material. Even though eddy currents have been investigated experimentally and theoretically, a theory of their abrupt formation is absent, particularly in the case when the tested iron samples are cylindrical wires. These processes play an important roll, whenever the so called magnetic viscosity (also known as the magnetic aftereffect) is experimentally determined.[1]

In the following, a theory of eddy-currents that form in a circular iron cylinder, which has its magnetization abruptly changed, is developed. The theory, which is in good agreement with experimental results, may be found to be of general application, for example, for the construction of fast acting electromagnets with massive or divided cores. As well, it is found that the starting equations are the same as those in the theory of heat conduction or diffusion.

[1]Here are meant the particular works that are concerned with the delay of spontaneous change in magnetization in the initial stages (investigation by J. Klemenčič, Weid. Ann. **62**, S. 68. 1897 and M. Gildemeister, Ann. D. Phys. **23**, S. 401, 1907). One could further elucidate this matter as indicated by the suggestion of W. Arkadiew as magnetic visocosity. With respect to this he suggests, under what is known as the magnetic after effect something like a change in magnetization, that expresses itself in small remaining changes in magnetization (investigation by J.A. Ewing, Phil trans. **176**, S. 589 19885; Lord Rayleigh, Phil. Mag. **23**, S. 225 1887). Whereas the initial processes occur very rapidly the after effect or magnetic viscosity occurs much slower (whole seconds, even minutes).

1. In this section, those mechanisms that delay the spontaneous change of magnetization in the initial stage are identified (Vgl. Fourier, Theorie anal. De la chaleur; also the article by E.W. Hobson and H. Diesselhorst in Teubners Enz. Math. Wiss. **5.** Nr. 4).

Description

H_ρ and B_ρ = Magnetic field intensity and induction at a distance ρ from the cylinder axis.
H_0 and B_0 = Initial values of these quantities at t=0.
σ = electrical conductivity (electromagnetic cgs units)
μ = permeability
j_ρ = Eddy current density at a distance ρ from the cylinder axis.
T= time
a = 4πμσ
r = radius of infinitiely long cylinder or ring.
$\xi = t/(ar^2)$
$J_0(x)$, $J_1(x)$, $J'_0(x)$, and $J'_1(x)$ = Bessel functions of argument *x* of the first kind of the zeroeth and first order and their derivatives with respect to *x*.

1. A round cylinder of iron with radius r is subjected to a longitudinally applied magnetic field H, that for t<0 has a constant field intensity H_0. However at t=0 this field is abruptly ended so that immediately the field becomes 0. Thereafter for t>0 within the iron only the field H_ρ is effective, producing eddy currents and the intensity H at a given t is only dependent on the distance ρ from the cylinder axis. Then the generally known relationships are:

$$H_\rho = 4\pi \int_\rho^r j_o d\rho \tag{1}$$

$$j_\rho = -\frac{\sigma}{4\pi\rho} \cdot \frac{\partial \Phi_\rho}{\partial t} = -\frac{\sigma}{4\pi\rho} \cdot \frac{\partial}{\partial t} \int_0^\rho 2\pi\rho \cdot B_\rho \cdot d\rho \tag{2}$$

After double differentiation with respect to ρ one obtains the differential equations that were put forward by Maxwell:

$$\frac{\partial^2 H_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial H_\rho}{\partial \rho} = 4\pi\rho \frac{\partial}{\partial t} B_\rho \tag{3}$$

$$-\frac{\partial^2 j_\rho}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial j_\rho}{\partial \rho} + \frac{j_\rho}{\rho^2} = \bar{\sigma} \frac{\partial^2}{\partial t \partial \rho} B_\rho \tag{4}$$

$$\frac{\partial^3 \vec{N}}{\partial \rho^3} - \frac{1}{\rho}\frac{\partial^2 \vec{N}}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial \vec{N}}{\partial \rho} = 4\pi\bar{\sigma} \frac{\partial^2}{\partial \rho \partial t} \Phi_o \tag{5}$$

These equations are generally applicable: but with the knowledge of the function B=B(H) they are difficult to deal with. However, if we are satisfied with the processes in the upper portion of the hysteresis loop, where one need consider only the disappearing portion of B_ρ for the various inductions one can set (W. Luthe, Verh. D. Deutsch. Phys. Ges **15.** S. 458. 1913).

$$B_\rho = \mu \cdot H_\rho;\ \mu = \text{const},$$

in this case simplifying the equations as:

$$\frac{\partial^2 B_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial B_\rho}{\partial \rho} = a\frac{\partial}{\partial t}B_\rho \tag{3′}$$

$$\frac{\partial^2 j_\rho}{\partial^2 \rho} + \frac{1}{\rho}\frac{\partial j_\rho}{\partial \rho} - \frac{j_\rho}{\rho^2} = a\frac{\partial}{\partial t}j_o \tag{4′}$$

$$\frac{\partial^3 \Phi_\rho}{\partial \rho^3} - \frac{1}{\rho}\frac{\partial^2 \Phi_\rho}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial \Phi_\rho}{\partial \rho} = a\frac{\partial^2}{\partial t \partial \rho}\Phi_\rho \tag{5′}$$

Zenneck, (J. Zenneck, Ann. D. Phys. **9**, S. 497, 1902) who for the first time applied equation (3′) to this field, has also determined the solution for the periodic case. We would like to find a solution that covers the case of spontaneous change in magnetization.

2. Therefore, we will first examine equation (3′). For the aperiodic process the particular integral result is given by the equation:

$$(B_\rho)_1 = A \cdot e^{-\frac{\alpha^2}{a}t} \cdot J_0(\alpha\rho)$$

Where A and α^2 for the time being are undetermined constants ($\alpha^2>0$). The complete integral will be given by the series:

$$B_\rho = \sum_{\nu=1}^{\nu=\infty} A_\nu \cdot e^{-\frac{\alpha_\nu^2}{a}t} \cdot J_0(\alpha_\nu \rho) \tag{6}$$

Now A_ν and α_ν^2 must be chosen so that the boundary conditions are met: The boundary conditions are:

1. For t=0 $B_\rho = B_0$ everywhere for $0 \le \rho \le r$.
2. For t>0 $B_\rho = 0$ $\rho = r$.

Following the theory of Bessel functions:

1. $\alpha_\nu r = \lambda_\nu$.

Where λ_ν is the ν^{th} root of the equation $J_0(\lambda) = 0$ and

$$2. \quad A_\nu = \frac{2\int_0^r B_0 J_0(\lambda_\nu \frac{\rho}{r})\rho d\rho}{r^2 J'(\lambda_\nu)^2} = \frac{2B_0}{\lambda_\nu J_1(\lambda_\nu)}$$

Setting this value for α_ν and A_ν in (6) one then obtains the formula for B_ρ as:

$$\frac{B_\rho}{B_0} = \beta_\rho = 2\sum_{\nu=1}^{\nu=\infty} e^{-\lambda_\nu^2 \xi} \cdot \frac{J_0(\lambda_\nu \frac{\rho}{r})}{\lambda_\nu J_1(\lambda_\nu)} \tag{7}$$

where

$$\xi = \frac{t}{4\pi\mu\sigma r^2}$$

is defined and the entire expression from the initial and boundary conditions is satisfied. As one can see, the variable ρ in this formula comes in only at the junction $\rho/r = x$, so that one can write $\beta_\rho = \beta_x$; $x=\rho/r$. The normalized values β_x and ξ now play the role of reduced coordinates; the difference x can now be examined as a parameter. Substituting in the values for λ_ν and $J_0(\lambda_\nu)$ in (7) we obtain:

$\beta_x = 1.60 \cdot e^{-5.78\xi} \cdot J_0(2.40\rho/r) - 1.07 \cdot e^{-30.47\xi} \cdot J_0(5.52\rho/r) + 0.85 \cdot e^{-74\xi} \cdot J_0(8.65\rho/r)$
$- 0.73 \cdot e^{-139\xi} \cdot J_0(11.79\rho/r) + 0.65 \cdot e^{-228\xi} \cdot J_0(14.93\rho/r) - 0.78 \cdot e^{-326\xi} \cdot J_0(18.07\rho/r)\ldots,$

An array, that of course rapidly converges for not too small a ξ ($\xi \geq 0.01$).
Figure 1 shows an array of curves for the variation of β_x for various ξ (for different times). The figure was generated with the help of Table 1 from the tables of Jahnke-Emdesche (E. Jahnke and F. Emde[1]) and Grunersche[2].

Table 1. Values of B_ρ

ξ \ ρ/r	0	0,2	0,4	0,6	0,8	0,9	1,0
0,01	1,0	1,0	1,0	0,98	0,81	0,49	0
0,03	1,00	1,00	0,977	0,866	0,534	0,280	0
0,09	0,885	0,850	0,737	0,541	0,279	0,137	0
0,27	0,337	0,318	0,263	0,183	0,090	0,044	0
0,81	0,0149	0,0140	0,0116	0,0081	0,0040	0,0019	0

[1] E. Jahnke and F. Emde, Funktionentafeln
[2]Grunersche, Jahrb. D. Radioakt. **3**, S. 120 1906.

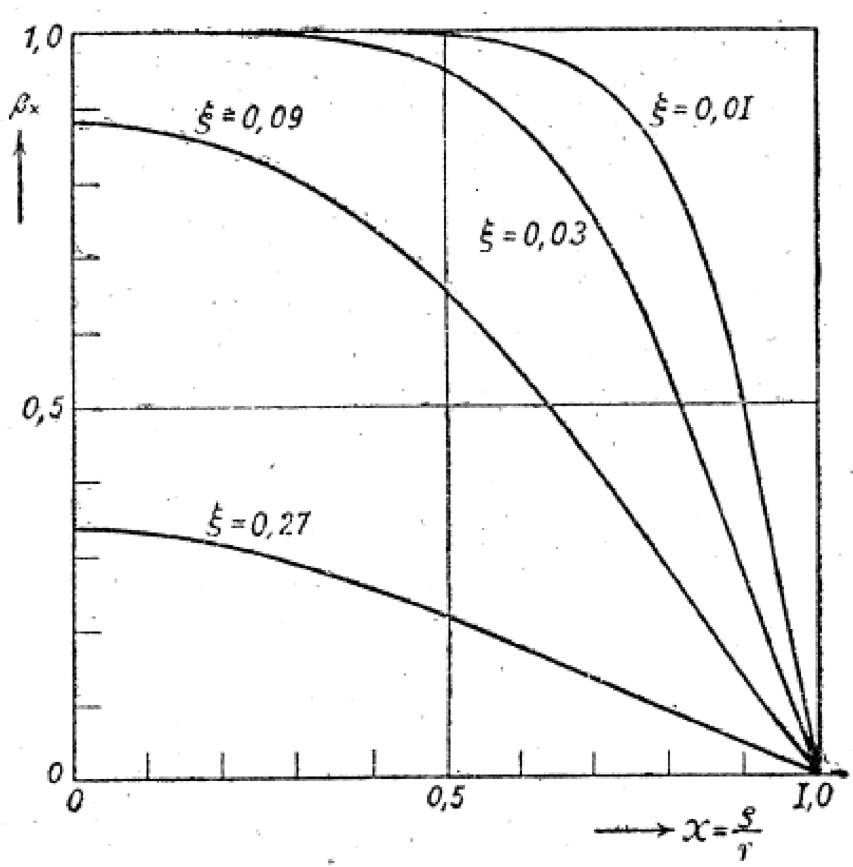


Fig. 1.

The demagnetization process begins on the outer surface of the cylinder and spreads itself gradually towards the centre. The thicker the cylinder, the larger its permeability and its conductance, the longer the process takes, analogous to the case of heat conduction or diffusion.

4. We now consider the particular case of the distribution of eddy currents. It is easy to see that for the distribution (4′) the expression:

$$j_\rho = \sum A_1 \cdot e^{-a^2 t} \cdot J_1(\alpha\rho) \tag{8}$$

is sufficient. To determine the arbitrary constants A_1 and α, we look at the expression:

$$j_\rho = -\frac{1}{4\pi\rho} \cdot \frac{\partial H_\rho}{\partial \rho} = -\frac{1}{4\pi\mu} \cdot \frac{\partial B_\rho}{\partial \rho} \tag{8′}$$

That follows from (1). Substituting in the array of (7) for B_ρ we obtain:

$$j_\rho = \frac{B_0}{4\pi\mu r} \sum_{\nu=1}^{\nu=\infty} e^{-\lambda_\nu^2 \xi} \cdot \frac{J_1(\lambda_\nu \frac{\rho}{r})}{J_1(\lambda_\nu)} \tag{8″}$$

Where ξ and λ have been previously defined. With these two arbitrary constants this expression is identical to (8), as well as being a solution to equation (4′). Differentiation of the infinite series (8″) proves the correctness of the solution. In other words the

equation (8″) is the sought after solution for the resulting distribution of eddy currents. The numerical formula for which is:

$$\frac{4\pi\mu r}{B_0} j_\rho = 1.93 \cdot e^{-5.78\xi} \cdot J_1(2.40\rho/r) - 2.94 \cdot e^{-30.47\xi} \cdot J_1(5.52\rho/r) + 3.68 \cdot e^{-74\xi} \cdot J_1(8.65\rho/r)$$
$$-4.30 \cdot e^{-139\xi} \cdot J_1(11.79\rho/r) + 4.84 \cdot e^{-228\xi} \cdot J_1(14.93\rho/r) - 5.32 \cdot e^{-326\xi} \cdot J_1(18.07\rho/r)\ldots,$$

With the help of this formula Table 2 is generated.

Table 2. Values of the function $(4\pi\mu r/B_0)j_\rho$ for various ξ.

ξ \ ϱ/r	0	0,2	0,4	0,6	0,7	0,8	0,9	1,0
0,01	0	0,0	0,0	0,0	0,32	0,72	1,29	2,54
0,03	0	0,016	0,106	0,499	0,828	1,16	1,37	1,37
0,09	0	0,161	0,385	0,584	0,662	0,704	0,704	0,660
0,27	0	0,095	0,173	0,221	0,234	0,235	0,227	0,210
0,81	0	0,004	0,008	0,010	0,011	0,011	0,011	0,005

As it is obvious from Figure 2, at first the entire eddy current density is concentrated on the outer surface of the cylinder and at t=0 $j_{\rho=r}$ is theoretically infinite[1]. Gradually the electric current penetrates towards the centre of the cylinder whereas the maximum density no longer lies on the outer surface.

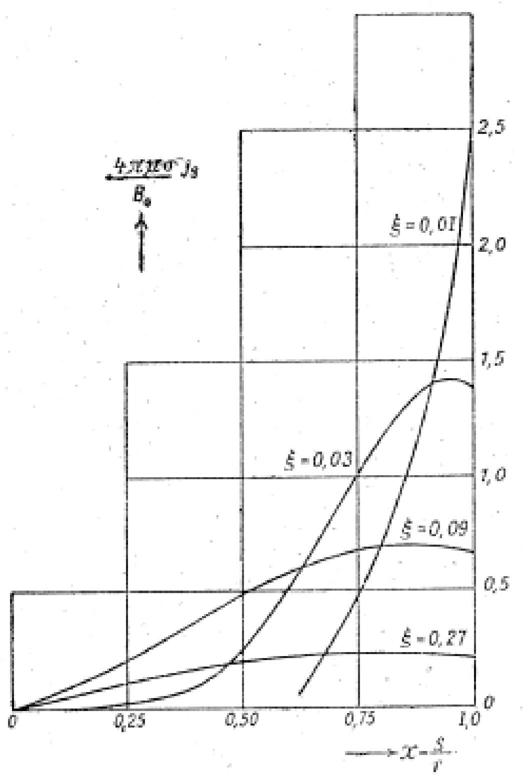


Fig. 2.

5. The two quantities B_ρ and j_ρ elude experimental investigation. The only thing that can be measured is the quantity Φ_ρ — the magnetic flux induced in the entire cylinder. The equation (5′) has an integral of the form:

$$\Phi_\rho = \rho r \sum A_2 \cdot e^{-a^2 \xi} \cdot J_1(\alpha\rho) \tag{9}$$

[1]Practically this is obviously not the case, first because the outer field doesn't immediately disappear and secondly since the iron possesses magnetic viscosity.

In analogy to the above, using the expression:

$$\Phi_\rho = \int_0^\rho 2\pi\rho \cdot B_\rho \cdot d\rho \tag{9′}$$

We obtain:

$$\Phi_\rho = 4\pi B_0 r\rho \sum_{\nu=1}^{\nu=\infty} e^{-\lambda_\nu^2 \xi} \cdot \frac{J_1(\lambda_\nu \frac{\rho}{r})}{\lambda_\nu^2 J_1(\lambda_\nu)} \tag{10}$$

Which satisfies equation (5′). Setting ρ=r and taking the ratio

$$\varphi = \frac{\Phi_r}{\pi r^2 B_0}$$

which reduces to the average induction, we obtain:

$$\varphi_\rho = 4 \sum_{\nu=1}^{\nu=\infty} \frac{e^{-\lambda_\nu^2 \xi}}{\lambda_\nu^2} \tag{11}$$

Therefore φ is a function of only the variable ξ. With the corresponding number formula read as:

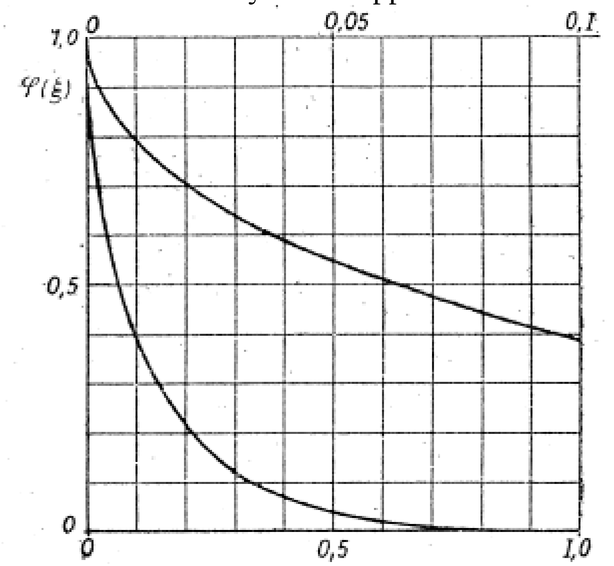
$$\varphi = 0.69167 \cdot e^{-5.783\xi} + 0.13120 \cdot e^{-30.472\xi} + 0.05346 \cdot e^{-74.83\xi} \cdot$$
$$+ 0.02773 \cdot e^{-139\xi} \cdot + 0.01794 \cdot e^{-228\xi} \cdot + 0.0122 \cdot e^{-326\xi} \ldots,$$

This series converges rapidly (particularly for not too small ξ) and is well suited for comparison. Table 3 shows the calculated values of φ.

Table 3. The reduced average induction φ(ξ).

ξ	φ(ξ)	ξ	φ(ξ)	ξ	φ(ξ)
0,01	0,7841	0,10	0,3855	0,40	0,0685
0,02	0,7016	0,15	0,2920	0,45	0,0513
0,03	0,6405	0,20	0,2180	0,50	0,0384
0,04	0,5905	0,25	0,1631	0,60	0,0216
0,06	0,5105	0,30	0,1221	0,70	0,0121
0,08	0,4473	0,35	0,0893	0,80	0,0068
				1,00	0,0021

Values of ξ for the upper curve.



Values of ξ for the lower curve.
Figure 3.

With the help of this table and in particular the help of figure 3 it is now easy to determine the fraction of the initial value of the average instantaneous induction that is examined at a t seconds, in a cylinder of any given thickness, conductivity, and permeability.

6. Since the equations (7), (8″) and (11), for the case of a decaying field, have been derived we can now easily obtain the solution for the spontaneous developing field where the boundary conditions are:

1. for t=0: $B_\rho = 0$ for $0 \le \rho \le r$,
   $B_\rho =$ const for $\rho = r$,
2. for $t = \infty$ $B_\rho =$ const for $0 \le \rho \le r$,

In accordance with the equations rearranged in the following way:
instead of equation (7):

$$\frac{B_\rho}{B_0} = \beta_\rho = 1 - 2\sum_{\nu=1}^{\nu=\infty} e^{-\lambda_\nu^2 \xi} \cdot \frac{J_0(\lambda_\nu \frac{\rho}{r})}{\lambda_\nu J_1(\lambda_\nu)} \tag{7′}$$

Instead of equation (8″):

$$j_\rho = -\frac{B_0}{4\pi\mu r}\sum_{\nu=1}^{\nu=\infty} e^{-\lambda_\nu^2 \xi} \cdot \frac{J_1(\lambda_\nu \frac{\rho}{r})}{J_1(\lambda_\nu)} \tag{8‴}$$

And finally instead of equation (11):

$$\varphi_\rho = 1 - 4\sum_{\nu=1}^{\nu=\infty} \frac{e^{-\lambda_\nu^2 \xi}}{\lambda_\nu^2} \tag{11′}$$

It is self evident how the reorganization of the tables and curves changes.

7. Examination of the experience of Hopkinson and Wilson[1], who, in the course of an abrupt change in magnetization in very thick (up to 10 cm) iron cylinders, found that the inductive relaxation times were proportional to the cross-sectional area of the cylinder. In the case that time is being measured the induction reduces to the x^{th} portion of its initial value, in other words that $\varphi(\xi)$ becomes 1/x at the same time, so we obtain, that the parameter ξ, depends only on 1/x, producing the equation:

$$\xi_{1/x} = \frac{1}{4\mu\sigma} \cdot \frac{\Theta_{1/x}}{\left(\pi r^2\right)}$$

or that $\Theta_{1/x}$ must really be proportional to the cylinder cross-section. Klemenčič[2] and Gildemeister[3] measured the viscosity in iron wires in the course of changing the temporary induction (within an arbitrary mass) as a function of time. In order to use their numbers to prove our formula we require a knowledge of the quantities σ and μ, for which exact instructions are missing, especially since the permeability μ does not agree at all with the usual permeability: First, since we are observing the upper proportion of the hysteresis loop, which corresponds to the instantaneous changes in magnetization, and secondly that the validity of the static value of their permeability is lost due to the great speed with which the process runs. From the beginning one can with good grounds maintain that our value of μ is decidedly smaller than the usual permeability. On closer examination we can't avoid concluding that the parameter μ is treated as an arbitrary constant in equation (11) and so choose them so that the equation is in best agreement with experiment. At the same time the average values of μ produce plausible numbers, typically 400, 200 and 100, which one can seen in the following table, Table 4.

Table 4. The Average Values of Permeability

Wire Thickness (mm)	6.0	1.74	0.50	0.36	0.185	0.161
Max field Strength (Gauss)	0.047	5.1	10	8.3	ca. 1	10
Permeability μ	(470)	145	116	180	(178)	360
Product $\mu\sigma$	0.047	—	—	—	0.0178	—

When the permeability is estimated (when the details concerning σ fail as well as the product $\mu\sigma$) every experimental curve can be reduced to produce $\varphi(\xi)$. The slope of such a curve is in agreement with equation (11) demonstrating its correctness. Figure 4 shows in one plot six experimental data sets together with four taken from unpublished works. As one sees, the theoretical curve is in good agreement with the generally observed data points and therefore appears to prove the derived formula.

[1]J. Hopkinson and E. Wilson, Proc. Roy. Soc. London **56**, pp. 108, 1894.
[2]Klemenčič
[3]Gildemeister

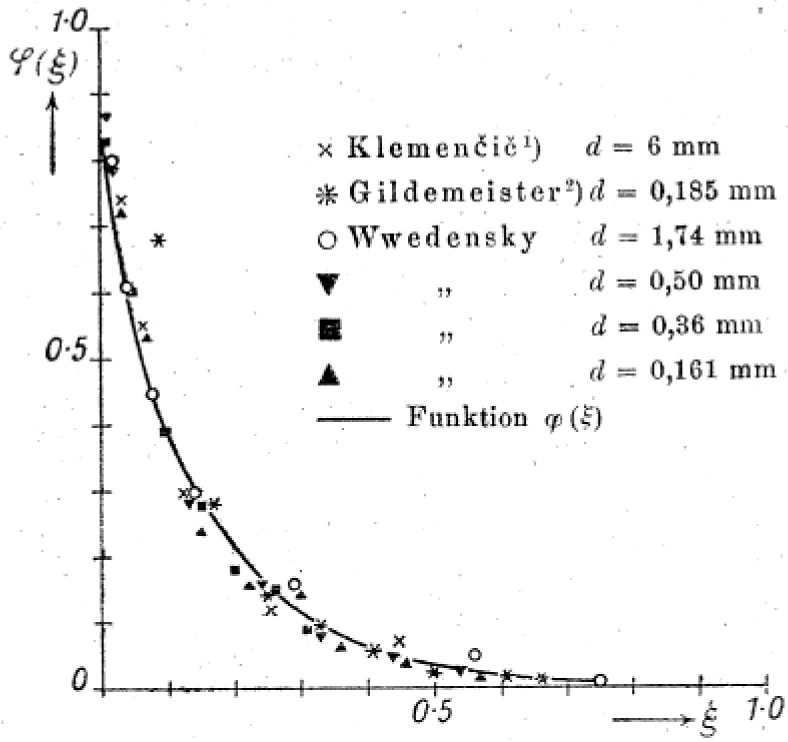


Fig. 4.

The agreement ceases when the wire becomes thinner than 0.1 mm. In this case a delay in the process, larger than what can be ascribed to eddy currents, is present. The induction falls almost to half of its initial value in about 10^{-6} sec., while the theory only predicts that only 10^{-7} sec. is required. Furthermore this delay, all things being equal, becomes more significant the larger the initial value of magnetization is. The reason for the delay is the magnetic viscosity, which regulates the course of spontaneous change in magnetization at its initial stages and in which I succeeded in my 1918 experiment to thoroughly investigate. I intend in the shortest possible time to talk on this subject.

I would like at this time to kindly thank Prof. W. Arkadiew for the stimulus and support provided for the realization of this work.

Moscow, August 1919.

1) Average of the experimental values.
2) Experimental h (corrected from the unloaded coils).

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[1]J. Hopkinson and E. Wilson, Proc. Roy. Soc. London **56**, S. 108 (1894).